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Spherically Symmetric Solutions of Einstein–Maxwell Theory with Null Fluid

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Abstract

Solutions of the Einstein–Maxwell equations with the addition of terms representing charged null fluid emitting from a spherically symmetric body are found. One type of solution is a simple extension of that found by Bonnor and Vaidya while the other represents a null electromagnetic field with null electric current.

1. *Introduction*

Bonnor & Vaidya (1970) have considered the Einstein–Maxwell theory with null fluid present and found a time-dependent spherically symmetric solution of the form

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + 2\,du\,dr + B\,du^2 \tag{1.1}$$

where

$$B = 1 - \frac{2m(u)}{r} + \frac{4\pi e^2(u)}{r^2}$$

The only non-zero component of the electromagnetic field tensor F_{ij} is

$$F_{14} = -F_{41} = \frac{e}{r^2}$$

The electric current vector, J^i, is null and is given by

$$J^i = -\frac{\dot{e}}{r^2}\delta_1{}^i$$

where $\dot{e} = de/du$. The null fluid current v^i is

$$v^i = \delta_1{}^i k$$

where

$$k^2 = \frac{1}{4\pi r^2}\left[-\dot{m} + \frac{4\pi e\dot{e}}{r}\right]$$

In this article we find all solutions of the form (1.1) for the field equations used by Bonnor and Vaidya which are

$$R_i^{\,j} - \tfrac{1}{2}\delta_i^{\,j} R = -8\pi(E_i^{\,j} + v_i v^j) \tag{1.2}$$

$$E_i^{\,j} = -F_{ia} F^{ja} + \tfrac{1}{4}\delta_i^{\,j} F^{ab} F_{ab} \tag{1.3}$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \tag{1.4}$$

$$F^{ij}{}_{;j} = J^i \tag{1.5}$$

$$v^i v_i = 0 \tag{1.6}$$

We no te that the non-zero components of $R_i^{\,j}$ are

$$R_1^{\,1} = R_4^{\,4} = -\tfrac{1}{2}\left(B'' + \frac{2}{r}B'\right)$$

$$R_2^{\,2} = R_3^{\,3} = \frac{1}{r^2}(1 - rB' - B)$$

$$R_4^{\,1} = -\frac{1}{r}\dot{B}$$

where the prime and the dot denote partial differentiation with respect to r and u respectively. From (1.2), (1.3) and (1.6) we find that the Ricci scalar vanishes which leads to the expression found by Bonnor and Vaidya, viz.

$$B = 1 - \frac{2m(u)}{r} + \frac{h(u)}{r^2}$$

where m, h are arbitrary functions.

2. *Solutions of the Field Equations*

Putting (i, j) respectively equal to (1,2), (1,3), (3,2), (4,2), (4,3) equations (1.2) and (1.3) give:

$$\frac{1}{r^2 \sin^2\theta} F_{13} F_{23} - F_{14} F_{21} + v_1 v_2 = 0 \tag{2.1}$$

$$\frac{1}{r^2} F_{12} F_{32} - F_{14} F_{31} + v_1 v_3 = 0 \tag{2.2}$$

$$BF_{12} F_{13} + F_{13} F_{24} + F_{12} F_{34} + v_2 v_3 = 0 \tag{2.3}$$

$$BF_{12} F_{41} - F_{41} F_{42} - \frac{1}{r^2 \sin^2\theta} F_{23} F_{43} + v_2 v_4 = 0 \tag{2.4}$$

$$BF_{13} F_{41} - F_{41} F_{43} - \frac{1}{r^2} F_{23} F_{42} + v_3 v_4 = 0 \tag{2.5}$$

Since $R_1^{\,1} = R_4^{\,4}$ we have

$$E_1^{\,1} + v_1 v^1 = E_4^{\,4} + v_4 v^4$$

which leads to

$$\frac{1}{r^2}F_{12}F_{12}+\frac{1}{r^2\sin^2\theta}F_{13}F_{13}+v_1v_1=0$$

so that

$$F_{12}=F_{13}=v_1=0 \tag{2.6}$$

Also $R_2{}^2=R_3{}^3$ gives

$$v_3v_3=v_2v_2\sin^2\theta$$

which, from (2.3) and (2.6), gives

$$v_2=v_3=0$$

The only non-zero equations are now (2.4) and (2.5) which become

$$F_{14}F_{24}r^2\sin^2\theta-F_{23}F_{34}=0$$
$$F_{14}F_{34}r^2+F_{23}F_{24}=0$$

From these two equations we see that there are two cases to consider.

Case I: $F_{24}=F_{34}=0$ and at least one of F_{14}, F_{23} non-zero.

Case II: $F_{14}=F_{23}=0$ and at least one of F_{24}, F_{34} non-zero.

Putting $v^1=v_4=k$ we find that the remaining field equations become

$$8\pi(F_{24}F_{24}+F_{34}F_{34}\operatorname{cosec}^2\theta+k^2r^2)=-2\dot{m}+\frac{\dot{h}}{r} \tag{2.7}$$

and

$$h=4\pi(F_{14}F_{14}r^4+F_{23}F_{23}\operatorname{cosec}^2\theta) \tag{2.8}$$

3. *Case I*

Case I is essentially the same as that discussed by Bonnor and Vaidya. If both F_{14} and F_{23} are non-zero then their solution is modified by the addition of a term representing a magnetic monopole, as in the static solution found by Hoffmann (1932). The function $h(u)$ is given by

$$h(u)=4\pi[e^2(u)+\sigma^2]$$

where σ is the magnetic pole strength and $e(u)$ is the total charge. The electromagnetic field components are

$$F_{14}=\frac{e(u)}{r}$$
$$F_{23}=\sigma\sin\theta$$

Note that from equation (1.4) σ is a constant.

The electric current and null fluid current are as given by Bonnor and Vaidya, namely

$$J^i = \left(-\frac{\dot{e}}{r^2}, 0, 0, 0\right)$$

and

$$v^i = (k, 0, 0, 0)$$

where

$$k^2 = \frac{1}{4\pi r^2}\left(-\dot{m} + \frac{4\pi e\dot{e}}{r}\right)$$

4. *Case II*

In this case the electromagnetic field is null. We find from equation (2.8) that $h = 0$ so that we have a Schwarzschild-type solution with metric

$$ds^2 = -r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + 2\, du\, dr + \left(1 - \frac{2m(u)}{r}\right) du^2 \tag{4.1}$$

Put $h = 0$ in (2.7) we see that $\dot{m} < 0$ so that the Schwarzschild mass is decreasing. We write

$$\alpha^2(u) = -\frac{\dot{m}}{4\pi} \tag{4.2}$$

so that (2.7) becomes

$$F_{24} F_{24} + F_{34} F_{34} \operatorname{cosec}^2\theta + k^2 r^2 = \alpha^2(u) \tag{4.3}$$

Equation (1.4) shows that

$$\frac{\partial F_{24}}{\partial r} = \frac{\partial F_{34}}{\partial r} = 0 \tag{4.4}$$

$$\frac{\partial F_{24}}{\partial \phi} = \frac{\partial F_{34}}{\partial \theta} \tag{4.5}$$

From (4.3) and (4.4) it follows that kr is independent of r, i.e.

$$k = \frac{p}{r}$$

where, in general, $p = p(\theta, \phi, u)$. It also follows from (4.4) that the only non-zero component of the electric current vector J^i is J^1 so that J^i is a null vector, i.e.

$$J^i J_i = 0$$

Thus in Case II the electromagnetic field is transverse to the direction of motion and both the field and the electric current are null. Unlike the solution found by Bonnor and Vaidya an injection of non-electromagnetic energy, supplied by the null field current, is not required to maintain the field. The power required per unit volume to maintain the given motion of charges in the field is zero and so is the power supplied per unit volume by the null fluid current.

The following are some possible solutions of equations (4.3), (4.4) and (4.5):

Solution (*a*)

$$F_{24} = \alpha(u)\cos\theta\cos\phi$$
$$F_{34} = \alpha(u)\sin\theta\sin\phi$$
$$J^i = -\frac{2\alpha(u)}{r^2}\sin\theta\cos\phi\,\delta_1{}^i$$
$$v^i = \frac{\alpha(u)}{r}\sin\theta\cos\phi\,\delta_1{}^i$$

Solution (*b*)

$$F_{24} = \beta(u)$$
$$F_{34} = 0$$
$$J^i = \frac{\beta(u)}{r^2}\cot\theta\,\delta_1{}^i$$
$$v^i = \frac{\gamma(u)}{r}\,\delta_1{}^i$$

where $\beta(u)$, $\gamma(u)$ are any two functions satisfying

$$\beta^2(u) + \gamma^2(u) = \alpha^2(u)$$

Solution (*c*)

$$F_{24} = \alpha(u)\sin\theta$$
$$F_{34} = 0$$
$$J^i = \frac{2\alpha(u)}{r^2}\cos\theta\,\delta_1{}^i$$
$$v^i = \frac{\alpha(u)}{r}\cos\theta\,\delta_1{}^i$$

Note that there is no solution for which $F_{24} = 0$, $F_{34} \neq 0$.

The metric (4.1) was found by Vaidya (1953) as a solution of the Einstein–Maxwell equations (without null fluid) together with F_{24}, F_{34} related by

$$F_{24}F_{24} + F_{34}F_{34}\operatorname{cosec}^2\theta = \alpha^2(u)$$

One possible set of values satisfying this relation is a special case of solution (b) corresponding to $\gamma(u) = 0$ so that $\beta(u) = \alpha(u)$. The inclusion of the null fluid results in a change in the value of J^i for this (and other) solutions and also gives rise to other solutions, such as solution (a), for which no analogous solutions exist when the null fluid is absent.

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